Feb. 10, 2014

Back to Differential Equations

What could $f(x)$ be?

- $\frac{d}{dx} f(x) = f(x)$ $\quad$ $(f(x) = e^x,\ f(x) = 2e^x, \ldots, f(x) = \underbrace{A}_{\text{Any Constant}} e^x)$
- $\frac{d}{dx} f(x) = -f(x)$

  $(f(x) = e^{-x},\ f(x) = Ae^{-x})$
- $\frac{d^2}{dx^2} f(x) = -f(x)$

  $(f(x) = \cos x,\ \sin x,\ \sin x + \cos x,\ A\sin x + B\cos x)$

How else can we write it?

- $\frac{d}{dx} f(x) = f(x)$ OR $\frac{dy}{dx} = y$ $\quad (y = Ae^x)$
- $\frac{d}{dx} f(x) = -f(x)$ OR $\frac{dy}{dx} = -y$ $\quad (y = Ae^{-x})$

we replace $f(x)$ with y.

Goal: Given an equation relating $x, y, y', y'', \ldots$ what is y?

Why do we care: We model behaviors in the "real world" with differential equations.

Examples: The rate of increase of a bacterial culture is proportional to the # of bacteria present at the time.

How do we write this w/ calculas?

$$\underbrace{\frac{dP}{dt}}_{\text{rate of increase of bacterial culture}} = \underbrace{k}_{\text{constant}}\underbrace{P}_{\text{\# of bacteria present}}$$

We'll come back to this stuff tomorrow. For now:

Solving differential Equations

(1) $\frac{dy}{dx} = f(x)$ $y =$ antiderivative of $f(x)$ $= F(x)$

(2) Guess and Check: Look at the problem and use what you know about derivatives to guess a solution.

Ex: $y' = \underbrace{3x^2 y}_{\text{chain rule will have been involved}}$ what are some solutions for y?

Check: $y = e^{x^2}$: $\frac{d}{dx} e^{x^3} = \underbrace{e^{x^3}}_{y} \cdot 3x^2 = 3x^2 \cdot y$ ✓

Seeing $\frac{dy}{dx}$: Slope Fields

Recall: there are infinitely many solutions to differential equations!

- $y = Ae^x$ solves $y' = y$
- $y = x^2 + C$ solves $y' = 2x$

Slope Field is a graph which helps us visualize solutions to a differential equation w/out us doing the solving.

Principle:

$$\frac{dy}{dx} = \underbrace{F(x,y)}_{\text{a function w/ x's, and y's}}$$

$$\left(\text{Example: } \frac{dy}{dx} = -\frac{x}{y} \right)$$

then you have a way of saying: "If I'm standing at point (a,b) then I should move from here with slope $F(a,b)$"

This works for:

$$\frac{dy}{dx} = -\frac{x}{y}$$

$$\frac{dP}{dt} = kP$$

$$\frac{dx}{dt} = t^2 \sin(xt) + x^2$$

Doesn't work for:

$$\frac{dy}{dx} = -\frac{x}{y} + \frac{d^2y}{dx^2}$$

$$\frac{dP}{dt} \cdot \frac{d^2P}{dt^2} = kP$$

(can't have other derivatives to calculate slopefield)

- Slope Field Worksheet

- Slope field applet:

  www.math.rutgers.edu/~sontag/JODE/JOde Applet.html

  Ex: $-\frac{x}{y} = \frac{dy}{dx}$

  Calculusapplets.com/slopefields.html

Look at the slope field for $\frac{dy}{dx} = -\frac{x}{y}$

When you draw a curve following the slope field, looks like semi-circles.

We know the equation of a semicircle is

$$y = \pm\sqrt{\underbrace{r^2}_{\text{a constant}} - x^2}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{r^2 - x^2}} \cdot -2x = \frac{-x}{\underbrace{\sqrt{r^2 - x^2}}_{y}} = -\frac{x}{y}$$

(so we used a slope field to guess a solution)

An Important "behind-the-scenes" thm

It's not immediately obvious that there is precisely one, unique sol'n to an IVP.

Peano's Existence Thm:

If $\frac{dy}{dx} = F(x,y)$ and $y(a) = b$ and $F(x,y)$ is continuous and defined on domain D then there is at least one diff'ble solution on the domain.

Uniqueness: If $F(x,y) = f(x)g(y)$ and g', f' are continuous, this solution is unique.